Concerning the Improper Definite Integral.[*]

By N. J. Lennes.

The following theorems suggested themselves in the course of writing out a detailed treatment of various topics in the elements of analysis. So far as known to the writer they are not, with one exception, to be found in the literature.

The aim of this note aside from pointing out that the improper definite integral possesses many of the important properties of the proper definite integral, is to show the impossibility of obtaining, in certain specified ways, stronger necessary conditions for the existence of the improper definite integral than have already been stated. It is, of course, no longer necessary to emphasize the importance of theorems to the effect that certain kinds of extensions of other theorems are impossible.

Definition : *A function is said to be bounded in the neighborhood of a point* $x = x_1$ *if there exists a pair of positive numbers* δ *and* M *such that for every* $x \mid |x - x_1| \leqq \delta$, $|f(x)| \leqq M$. *The function is unbounded in the neighborhood of a point if no such pair of numbers exist.*

If (a) a function $f(x)$ considered in the interval $a \ldots . b$ is bounded in the neighborhood of every point on this interval except $x = a_1$,

(b) $\int_x^b f(x)\, dx$ exists for every value of x on the segment $a \ldots . b$[†], (the segment not including its end-points),

(c) $\underset{x \doteq a}{\angle} \int_x^b f(x)\, dx$ exists and is finite,

then the improper definite integral is said to exist in the interval $a \ldots . b$, or the integral is said to exist improperly at $x = a$.

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† The definite integral is said to exist only for functions which are bounded on the interval under consideration. Hence (a) is redundant in the presence of (b).

Lemma: *For every function $f_1(x)$ which is unbounded in the neighborhood of $x=a$ there is a function $f_2(x)$ which is infinitesimal as x approaches a* such that $f_1(x) \cdot f_2(x)$ is unbounded in the neighborhood of this point.* (It is to be understood that the functions $f_1(x)$ and $f_2(x)$ are considered only for values of x in an interval $a \dots b$, *i. e.* on one side of the point a.)

Proof: Decompose the interval $a \dots b$ into an infinite set of subintervals by means of an ordered set of points dense only at a.† Let this partition of $a \dots b$ be such that for some given positive number M there are values of x in the interval $x_1 \dots b$ for which $|f_1(x)| > 2M$ and in general there are values of x in $x_n \dots x_{n-1}$ for which $|f_1(x)| > 2^n M$. We define $f_2(x)$ as follows: $f_2(x) = 1$ for all values of x in the interval $x_1 \dots b_1$, $f_2(x) = \frac{1}{2}$ for all values of x in $x_2 \dots x_1$ excluding the point x_1, $f_2(x) = \frac{1}{n}$ in $x_n \dots x_{n-1}$ excluding the point x_{n-1}. Then there are values of x in $x_n \dots x_{n-1}$ for which $|f_1(x)| \cdot f_2(x) > \frac{2^n}{n} M$ which shows $f_1(x) \cdot f_2(x)$ unbounded in the neighborhood of $x=a$. Evidently $f_2(x)$ may be so chosen that $\frac{f_2(x)}{x-a}$ shall be monotonic increasing as x approaches a.‡

THEOREM I. *For every function $f_1(x)$ which is unbounded in the neighborhood of $x=a$ there exists a non-oscillating* § *function $f_2(x)$ such that $\underset{x \doteq a}{\angle} \int_x^b f_2(x)dx$ exists and is finite while $(x-a) f_1(x) \cdot f_2(x)$ is unbounded in the neighborhood of $x=a$.*

Proof: According to the lemma there exists a function $f_3(x)$ such that $\underset{x \doteq a}{\angle} f_3(x) = 0$ while $f_3(x) \cdot f_1(x)$ is unbounded. Consider a function $f_4(x) = \frac{f_3(x)}{x-a}$, f_4 being monotonic as x approaches a. Since $(x-a) f_4(x) \cdot f_1(x) = f_3(x) \cdot f_1(x)$, $(x-a) f_4(x) \cdot f_1(x)$ is unbounded in the neighborhood of $x=a$. Let $x_1 \dots x_n \dots$

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*$f(x)$ is infinitesimal as x approaches a if $\underset{x \doteq a}{\angle} f(x) = 0$.

† A set of points is said to be dense at a certain point a if there are points of the set, other than a, within every neighborhood of a.

‡ In case $f_1(x) > 0$ for all values of x in same neighborhood of $x=a$ $f_2 = \frac{1}{\sqrt{f_1(x)}}$ or $f_2 = \frac{1}{\log f_1(x)}$ would satisfy the conditions imposed upon $f_2(x)$ by the lemma.

§ A function $f(x)$ is said to be non-oscillating in an interval if as x increases in this interval the function either never decreases or never increases. This is different from a monotonic function which either constantly increases or constantly decreases.

be an ordered series of points in $a \ldots . b$ dense only at a such that $f_3(x) . f_1(x)$ is unbounded at this set. Then in the sequence

$$(x_1 - a) f_4(x_1), \quad (x_2 - a) f_4(x_2), \ \ldots . \ (x_n - a) f_4(x_n) \ \ldots . \qquad (1)$$
$$\underset{n \doteq \infty}{\angle} (x_n - a) f_4(x_n) = 0 \text{ since } \underset{x \doteq a}{\angle} (x - a) f_4(x) = 0.$$

Hence there is a value of n, n_1 such that

$$|(x_1 - a) f_4(x_1)| \geqq 2 |(x_{n_1} - a) f_4(x_{n_1})|$$

and another value of n, n_2 such that

$$|(x_{n_1} - a) f_4(x_{\ 1})| \geqq 2 |(x_{n_2} - a) f_4(x_{n_2}).$$

In general n_{m+1} is so chosen that

$$|(x_{n_m} - a) f_4(x_{n_m})| \geqq 2 |(x_{n_{m+1}} - a) f_4(x_{n_{m+1}})|.$$

In this manner we select from the sequence (1) a set of terms forming the convergent series:

$$(x_1 - a) f_4(x_1) + (x_{n_1} - a) f_4(x_{n_1}) + \ \ldots . \ + (x_{n_m} - a) f_4(x_{n_m}) + \ \ldots . \qquad (2)$$

We then obtain a function $f_2(x)$ as follows:

For the set of values of x, $x_{n_{m+1}} < x \leqq x_{n_m}$,

$$f_2(x) = |f_4(x_{n_m})|.$$

Then

(a) $f_2(x)$ is non-oscillating since $|f_4(x_{n_m})| < |f_4(x_{n_{m+1}})|$,

(b) $(x - a) \ f_2(x) . f_1(x)$ is unbounded at the set $x_1, x_{n_1}, x_{n_2} \ldots . x_{n_m} \ldots .$ since for this set $f_2(x) . f_1 \equiv f_4(x) . f_1(x)$,

(c) $\underset{x = a}{\angle} \int_x^b f_2(x) dx = \sum_{m=0}^{\infty} (x_{n_{m+1}} - x_{n_m}) f_4(x_{n_m})$.

But the terms of this series are numerically smaller than the corresponding terms of the convergent series (2). Hence $\underset{x \doteq a}{\angle} \int_x^b f_2(x) dx$ exists and is finite.

In case $f_2(x)$ is non-oscillating and unbounded in the neighborhood of $x = a$ a well-known necessary condition that $\underset{x \doteq a}{\angle} \int_x^b f_2(x) dx$ shall exist and be finite is that $\underset{x \doteq a}{\angle} (x - a) f_2(x) = 0$. Theorem I may be regarded as showing this a strong necessary condition since according to it no function $f_1(x)$ of x can be pre-assigned which shall approach infinity slowly enough as x approaches a so that

$$\underset{x \doteq a}{\angle} (x-a) f_1(x) \, . \, f_2(x) = 0.*$$

Lemma: *If* $\int_a^b f_1(x)dx$ *and* $\int_a^b f_2(x)dx$ *both exist, and if in the interval* $a \ldots . b$ $\left| \frac{f_1(x)}{f_2(x)} \right| \leqq M$ *and* $f_2(x)$ *does not change sign, then*

$$\left| \int_a^b f_1(x)dx \right| \leqq M \left| \int_a^b f_2(x)dx \right| .$$

Proof: From the definition of the definite integral and the obvious proposition that if $n_0, n_1 \ldots . n_n$ are numbers having the same sign and if

$$|v_0| \leqq |n_0|, \quad |v_1| \leqq |n_1|, \quad \ldots\ldots\ldots, \quad |v_n| \leqq |n_n|$$

then

$$\left| \sum_{i=1}^{n} v_i \right| \leqq \left| \sum_{i=1}^{n} n_i \right|$$

it follows that

$$\int_a^b |f_1(x)| \, dx \leqq M \int_a^b |f_2(x)| \, dx.$$

But

$$\left| \int^b f_1(x)dx \right| \leqq \int_a^b |f_1(x)| \, dx \text{ and } \int_a^b |f_2(x)| \, dx = \left| \int_a^b f_2(x)dx \right|$$

therefore

$$\left| \int_a^b f_1(x)\, dx \right| \leqq M \left| \int_a^b f_2(x)\, dx \right| .$$

THEOREM II. *If (a)* $\frac{f_1(x)}{f_2(x)}$ *is bounded in some neighborhood of* $x = a$,

(*b*) $\int_x^b f_1(x)dx$ *and* $\int_x^b f_2(x)dx$ *both exist for every* x *in the segment* $a \ldots . b$,†

(*c*) $f_2(x)$ *does not change sign in some neighborhood of* $x = a$,

(*d*) $\underset{x \doteq a}{\angle} \int_x^b f_2(x)dx$ *is finite,*‡

then it follows that $\underset{x \doteq a}{\angle} \int_x^b f_1(x)dx$ *exists and is finite.*

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*This theorem was, in effect, stated and proved by A. Pringsheim in Mathematische Annalen, vol. 37 (1890), p. 591. The proof given here is however much shorter and simpler than that given by Pringsheim.

† The expression segment $a \ldots . b$ is used to include all values of x, $a < x < b$.

‡ We notice that since under (c) of the hypothesis $f_2(x)$ does not change sign $\underset{x \doteq a}{\angle} \int_x^b f_2(x)dx$ cannot *fail* to exist, finite or infinite, for it follows from this that $\int_x^b f_2(x)dx$ is a non-oscillating function of x whence by a well-known theorem the limit exists.

Proof: Since by hypothesis $\underset{x \doteq a}{\angle} \int_x^b f_2(x)dx$ exists and is finite it follows by an elementary theorem that for every ε there exist a δ_ϵ such that for every x_1 and x_2 on the segment $a \ldots . b$ and also in a δ_ϵ neighborhood of $x = a$,

$$\left| \int_{x_1}^{x_2} f_2(x)dx \right| < \varepsilon .$$

Consider x_1 and x_2 in a neighborhood of $x = a$ in which $\left| \frac{f_1(x)}{f_2(x)} \right| \leqq M$ and $f_2(x)$ does not change sign.

Then by the lemma:

$$\left| \int_{x_1}^{x_2} f_1(x)dx \right| \leqq M \left| \int_{x_1}^{x_2} f_2(x)dx < M \varepsilon. \right.$$

Since $M\varepsilon$ can be made small at will by making ε small (M being independent of ε) it follows that $\underset{x \doteq a}{\angle} \int_x^b f_1(x)dx$ exists and is finite. This may be called the relative convergence theorem.

Definition: *If* $\frac{f_1(x)}{f_2(x)}$ *and* $\frac{f_2(x)}{f_1(x)}$ *are both bounded in the neighborhood of* $x = a_1$, $f_1(x)$ *and* $f_2(x)$ *being unbounded in that neighborhood, then we refer to these functions as having the same rank of infinity in the neighborhood of* $x = a$. *If* $\underset{x \doteq a}{\angle} \frac{f_1(x)}{f_2(x)} = K$, $K \neq 0$, *then they are said to be of the same order of infinity.*

THEOREM III. *If (a)* $f_1(x)$ *and* $f_2(x)$ *are of the same rank of infinity in the neighborhood of* $x = a$,

(b) $\int_x^b f_1(x)dx$ *and* $\int_x^b f_2(x)dx$ *both exist for every* x *on the segment* $a \ldots . b$,

(c) there is a neighborhood of $x = a$ *in which* $f_2(x)$ *does not change sign,*

(d) $\underset{x \doteq a}{\angle} \int_x^b f_2(x)$ *is infinite, (see note under Theorem II)*

then $\underset{x \doteq a}{\angle} \int_x^b f_1(x)dx$ *cannot exist and be finite.*

Proof: This is a direct consequence of theorem II since by that theorem if $\underset{x \doteq a}{\angle} \int_x^b f_1(x)dx$ were finite then $\underset{x \doteq a}{\angle} \int_x^b f_2(x)dx$ would exist and be finite.

THEOREM IV. *If for a function $f_1(x)$ which does not change sign in the neighborhood of $x=a$ there exists a monotonic function $f_2(x)$ such that $f_1(x)$ and $f_2(x)$ have the same rank of infinity in the neighborhood of $x=a$, $\int_x^b f_1(x)dx$ and $\int_x^b f_2(x)dx$ both existing for any x on the segment $a \ldots . b$, then a necessary condition that $\angle_{x \doteq a} \int_x^b f_1(x)dx$ exists and is finite is that $\angle_{x \doteq a} (x-a) f_1(x) = 0$.*

Proof: By hypothesis $\angle_{x \doteq a} \int_x^b f_1(x)dx$ exists and is finite. Hence by theorem II $\angle_{x \doteq a} \int_x^b f_2(x)dx$ exists and is finite. Therefore by an elementary theorem (referred to on page 17) $\angle_{x \doteq a} (x-a) f_2(x) = 0$.

Since $\dfrac{f_1(x)}{f_2(x)}$ is bounded as x approaches a, *i. e.*, there exists an M such that $|f_1(x)| < M|f_2(x)|$, we have $|(x-a)|f_1(x)| < M|(x-a) f_2(x)|$.

But
$$\angle_{x \doteq a} M|(x-a) f_2(x)| = 0.$$
Therefore
$$\angle_{x \doteq a} |(x-a) f_1(x)| = 0 \text{ is } \angle_{x \doteq 0} (x-a) f_1(x) = 0.$$

THEOREM V. *For every function $f_1(x)$ defined in the interval $a \ldots\ b$ there exists a function $f_2(x)$ such that*

(a) $f_2(x)$ is continuous and does not change sign in a certain neighborhood of $x=a$,

(b) $\angle_{x \doteq a} \int_x^b f_2(x)dx$ exists and is finite,

(c) For x on a certain set $[x']$ $\angle_{x' \doteq a} \dfrac{f_1(x')}{f_2(x')} = 0$.

Proof: (We describe the function $f_2(x)$ in the language of geometry).

Let $x_1', x_2', \ldots . x_n' \ldots .$ be a set of points in the interval $a \ldots . b$ dense only at a. Let $B_1, B_2 \ldots . B_n \ldots .$ be a set of numbers such that

$$B_n \,.\, n \,.\, f_1(x_n) \geqq 2\ B_{n+1} \,.\, (n+1) \,.\, f_1(x_{n+1})\ (n = 1 \ldots . \infty).$$

On the x axis lay off segments $A_n\ C_n = B_n$* such that x_n are the middle points of the segments $A_n\ C_n$. The segments $A_n\ C_n$ must satisfy the further condition

*There is no incongruity in speaking of segments equal to numbers if the statement is properly interpreted.

that they shall be small enough so as not to overlap. (This condition can be satisfied by making the numbers B_n small enough.) On the segments $A_n C_n$ as bases construct isosceles triangles on the positive side of the x-axis whose altitudes are $n \,.\, f_1(x_n)$. The measures of area of these triangles form a convergent series. Let $f_3(x)$ be a continuous function, monotonic and unbounded in the neighborhood of $x = a$, such that $\underset{x \doteq a}{\angle} \int_x^b f_3(x)dx$ exists and is finite. $f_3(x)$ should also satisfy the condition that the curve which it represents has only one point in common with a side of one of the triangles described above. We then define $f_2(x)$ as the function represented by the following curve: 1st, those parts of the boundaries of the isosceles triangles just described which lie above the curve defined by $f_3(x)$. 2d, those parts of the curve defined by $f_3(x)$ which lie outside these triangles or on their boundary.

Obviously the function so defined has the properties specified in the theorem. From theorem V it follows that from the hypothesis only that $\underset{x \doteq a}{\angle} \int_x^b f_1(x)dx$ exists and is finite, it is impossible to obtain any conclusion as to the boundedness of any function whatever $f_2(f_1(x))$ provided the values of f_2 depend only upon the values of $f_1(x)$ at a set of points whose limit points form a discrete closed set, and provided also f_2 is capable of becoming infinite for any values however large of the independent variable $f_1(x)$ at such points.

This is precisely what one would expect *a priori*, since the definite integral is a function of two dimensions while any condition in terms of boundedness of a function would necessarily be in terms of only one of these.

In case the definite integral exists *improperly* at a finite number of points and properly at all other points, the theorems just stated apply directly. In case the points at which the definite integral exists improperly are dense at a point a but discrete at all other points of a neighborhood including this point, we say the definite integral exists improperly at $x = a$ if $\underset{x \doteq a}{\angle} \int_x^b f(x)dx$ exists and is finite, the limitand function being the improper definite integral in the interval $a \ldots . b$. Hereafter the sign $\int$ will include the improper as well as the proper definite integral, the term improper definite integral being used to denote an integral existing improperly at a set of points whose limit points form a discrete closed set.

The following theorems on the definite integral apply to the improper definite integral as well:

THEOREM VI. *If* $\int_a^b f(x)dx$ *exists then* $\int_a^c f(x)dx$ *and* $\int_c^b f(x)dx$ *both exist and* $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

The proof is obvious.

THEOREM VII. *If* $\int_a^b f(x)dx$ *exists then* $\int_a f(x)dx$ *is a continuous function of* x *for all values of* x *in the interval* $a \ldots . b$.

Proof: The extension of the theorem to the improper definite integral follows immediately from the definition of the latter.

THEOREM VIII. *If* (*a*) $\int_a^b f_1(x) \,.\, f_2(x)dx$ *and* $\int_a^b f_1(x)$ *both exist,*

(*b*) $f_1(x)$ *does not change sign in* $a \ldots . b$,

(*c*) $\overline{B}$ *and* $\underline{B}$ *are the least upper and the greatest lower bound respectively of* $f_2(x)$ *in* $a \ldots . b$,

then
$$\overline{B} \int_a^b f_1(x) \,.\, dx \leqq \int_a^b f_1(x) \,.\, f_2(x)dx \leqq \overline{B} \int_a^b f_1(x) \quad \text{or}$$

$$\underline{B} \int_a^b f_1(x)dx \geqq \int_a^b f_1(x) \,.\, f_2(x)dx \geqq \overline{B} \int_a^b f_1(x)dx$$

Proof: The theorems $\overline{B} \int_a^b f_1(x)dx = \int_a^b \overline{B} f_1(x)dx$ and

$$\underline{B} \int_a^b f_1(x)dx = \int_a^b \underline{B} f_1(x)dx$$

$\overline{B}$ and $\underline{B}$ being constants, apply to the improper definite integral as well as to the proper since $\underset{x \doteq x_1}{\angle} af(x) = a \underset{x \doteq x_1}{\angle} f(x)$ when a is a constant.

Moreover in case $f_1(x)$ is always positive

$$\overline{B} f_1(x) \leqq f_1(x) \,.\, f_2(x) \leqq \overline{B} f_1(x).$$

Hence by the definitions of the definite integral and elementary considerations:

$$\int_a^b \underline{B} f_1(x)dx \leqq \int_a^b f_1(x)\, f_2(x)dx \leqq \int_a^b \overline{B} f_1(x)dx,$$

and therefore

$$B \int_a^b f(x)dx \leqq \int_a^b f_1(x)\, f_2(x)dx \leqq \overline{B} \int_a^b f_1(x)dx.$$

If $f_1(x)$ is entirely negative it follows in the same manner that

$$\underline{B} \int_a^b f_2(x)dx \geqq \int_a^b f_1(x) \cdot f_2(x)dx \geqq \overline{B} \int_a^b f_1(x).$$

COROLLARY. (The first mean value theorem).

If in theorem VIII $f_2(x)$ is continuous there is a value of x, ξ in the interval $a \ldots . b$ such that

$$\int_a^b f_1(x) \cdot f_2(x)dx = f(\xi) \cdot \int_a^b f_1(x)dx.$$

THEOREM IX. (The second mean value theorem).

If the improper definite integral of $f_1(x)$ exists in $a \ldots . b$ and if $f_2(x)$ is non-oscillating in this interval, then there is a value of x, $x = \xi$ in the interval $a \ldots . b$ such that

$$\int_a^b f_1(x) \cdot f_2(x)dx = f_2(a) \int_a^{\xi'} f_1(x)dx + f_2(b) \int_{\xi'}^b f_1(x)dx,$$

Proof: If $f_2(x)$ is non-oscillating it is integrable, consequently $\int_a^b f_1(x) . f_2(x)dx$ exists since the product of two integrable functions, one being integrable in the proper sense, can readily be shown to be integrable at least in the improper sense.

Suppose the integral of $f_1(x)$ to exist properly at every point of the interval $a \ldots . b$ except the point c. Consider a function $f_3(x)$ defined as follows in the interval $a \ldots . b$. $f_3(x) = f_2(x)$ in the interval $a \ldots . c - e$ (e being a small positive number) $f_3(x) = 0$ in the interval $c - e \ldots . c + e$ and $f_3(x) = f_1(x)$ in the interval $c + e \ldots . b$.

Assuming theorem IX where the integral exists in the proper sense we have that a ξ exists for every value of e however small such that

$$\int_a^b f_3(x) \cdot f_1(x)dx = f_3(x) \int_a^{\xi} f_1(x)dx + f_3(b) \int_{\xi}^b f_1(x)dx.$$

As e approaches zero, ξ approaches one or more points as limit points. Let ξ' be such limit point. Then we assert

$$\int_a^b f_2(x) \cdot f_1(x)dx = f_2(a) \int_a^{\xi'} f_1(x)dx + f_2(b) \int_{\xi'}^b f_1(x)dx. \quad (1)$$

Suppose this equation does not hold and that

$$\left| \int_a^b f_2(x) \cdot f_1(x)dx - f_2(a) \int_a^{\xi'} f_1(x)dx - f_2(b) \int_{\xi'}^b f_1(x)dx \right| = \varepsilon. \quad (2)$$

Since the definite integral is a continuous function of the limits of integration it follows that there exists a certain neighborhood of $x = \xi'$ such that for every value of x in this neighborhood

$$\left| f_2(a) \int_a^{\xi'} f_1(x)dx + f_2(b) \int^b f_1(x)dx - f_2(a) \int_a^x f(x)dx - f_2(b) \int_x^b f_1(x)dx \right| < \frac{\varepsilon}{2}.$$

Denote this neighborhood by δ_ϵ.

Likewise there is a certain neighborhood of $x = c$ which we denote by δ'_ϵ such that

$$\left| \int_{\epsilon + \delta_\epsilon}^{\epsilon + \delta_\epsilon} f_1(x) \cdot f_2(x)dx \right| < \frac{\varepsilon}{2} \cdot$$

Therefore for all values of $\varepsilon < \delta_\epsilon$ the corresponding values of ξ cannot lie in the δ_ϵ neighborhood of $x = \xi'$ and consequently ξ' is not a value approached by ξ as e approaches zero, which is a contradiction with the hypothesis resulting from the assumption of equation (2). Hence equation (1) holds. In this manner we can prove the existence of a ξ' satisfying the conditions of the theorems if the definite integral of $f_1(x)$ exists properly everywhere in the interval except at a finite number of points. If the integral exists improperly at an infinite set of points whose limit points (first derived set) is a closed discrete set and if the improper definite integral as defined on page 19 exists in the interval $a \ldots . b$ then the argument to show the existence of ξ' for such function is identical with the above.

The restriction on the set of points at which the integral exists improperly is rather artificial. Obviously the theorems can be extended to cover more general cases.

NOTE: For a more detailed discussion of the elementary properties of the improper definite integral see Infinitesimal Analysis (pp. 191-224), Veblen and Lennes. Published by John Wiley & Sons, New York. This book also contains a treatment of the improper definite integral in the case where it exists improperly at a more general set of points.

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