LIBOR Rate Model

LIBOR Rate Model is used for pricing Libor-rate based derivative securities. The model is applied, primarily, to value instruments that settle at a Libor-rate reset point. In order to value instruments that settle at points *intermediate* to Libor resets, we calculate the numeraire value at the settlement time by interpolating the numeraire at bracketing Libor reset points.

Libor rate model is very useful to price callable exotics. Many derivatives have callable features. Callable exotics are among the most challenging derivatives to price. These products are loosely defined by the provision that gives the holder or issuer the right to call the product after a lock-out period (more details at https://finpricing.com/lib/EqCallable.html).

Let L denote a Libor rate that sets at time T for an accrual period Δ. A European caplet on L is an option with payoff at $T+\Delta$ of the form

$$\Delta \max\left(L - X, 0\right).$$

Similarly, a floorlet has payoff of the form

$$\Delta \max\left(X - L, 0\right).$$

Consider a European option on a fixed-for-floating-rate swap specified by

- forward start time, T,
- set of future resets, $\{T_i\}$, where $T < T_1 < ... < T_n$,
- floating-leg payments of ΔL at T_{i+1} where L denotes the spot Libor at T_i for the accrual period $\Delta = T_{i+1} - T_i$.

A European *payer* swaption has payoff at T of the form

$$\sum_i \Delta P(T,T_i)\max(\kappa - X, 0)$$

where

$$\kappa = \frac{1 - P(T,T_n)}{\sum_i \Delta P(T,T_i)}$$

is the spot swap rate at time T. A European *receiver* swaption has payoff at T of the form

$$\sum_i \Delta P(T,T_i)\max(X - \kappa, 0).$$

Consider a set of future Libor reset dates, $\{T_i\}$, where $0 < T_1 < ... < T_n$. Let $L_i(t)$ denote the forward Libor rate, as seen at time t, which sets at time T_i for the accrual period $\delta_i = T_{i+1} - T_i$. We seek to model Libor rates under the spot Libor measure, which has numeraire process,

$$N_{T_i} = \prod_{j=1}^{i} \frac{1}{P(T_{j-1}, T_j)},$$

where $P(t,T)$ denotes the price at time t of a zero coupon bond with maturity of T.

Let $[t]$ denote the integer, i, such that $T_{i-1} < t \le T_i$. Under the spot Libor measure, WM assumes that

$$dL_i = L_i\, \vec{\sigma}(t,T_i) \bullet \left(\sum_{j=[t]}^{i} \frac{\delta_j L_j(t)}{1 + \delta_j L_j(t)} \vec{\sigma}(t,T_j) dt + d\vec{W} \right) \tag{3.1}$$

for $i = 1, \ldots, n-1$, where

- $\vec{W} \in \Re^4$ is a vector of uncorrelated, standard Brownian motions,
- $\vec{\sigma} \in \Re^4$ is a time deterministic volatility vector, which we define below.

We primarily consider interest rate derivatives that depend on the set of Libor rates above, $\{L_i\}_{i=1}^{n-1}$, and that settle at one of the reset times above, $\{T_i\}_{i=1}^{n}$. Consider, for example, a payoff, X, at time T_i. This payoff then has value

$$E\left(\frac{X}{N_{T_i}}\right).$$

A volatility vector is of the form

$$\vec{\sigma}(t,T) = \vec{\psi}(T-t)\upsilon(t,T).$$

Here

$$\begin{aligned}
\psi_1(t) &= c, \\
\psi_2(t) &= \sqrt{\frac{1-c^2}{1+(t/a)^{\alpha}}}, \\
\psi_3(t) &= \sqrt{\left(1-c^2\right)\left(1-\frac{1}{1+(t/a)^{\alpha}}\right)\left(\frac{1}{1+(t/a)^{\beta}}\right)}, \\
\psi_4(t) &= \sqrt{1-[\psi_1(t)]^2-[\psi_2(t)]^2-[\psi_3(t)]^2}.
\end{aligned}$$

Furthermore

$$\upsilon(t,T) = e^{\sum_i \sum_j a_{ij} \Phi_i(x_1) \Phi_j(x_2)}$$

where

$$x_1 = -1 + 2e^{-(T-t)/\tau_1},$$
$$x_2 = -1 + 2e^{-T/\tau_2}.$$

Here

$$\Phi_i(x) = \cos\left(i \cos^{-1} x\right)$$

denotes a Chebyshev polynomial of the first kind.

In the above, the parameters $c, \alpha, \beta, \tau_1, \tau_2$ and $\{a_{ij}\}$ are determined from calibration.